Fundamental Frequency of Continuous Signals

To identify the period T, the frequency $f = 1/T$, or the angular frequency $\omega = 2\pi f = 2\pi/T$ of a given sinusoidal or complex exponential signal, it is always helpful to write it in any of the following forms:

$$sin(\omega t) = sin(2\pi f t) = sin(2\pi t/T)$$

The fundamental frequency of a signal is the greatest common divisor (GCD) of all the frequency components contained in a signal, and, equivalently, the fundamental period is the least common multiple (LCM) of all individual periods of the components.

Example 1: Find the fundamental frequency of the following continuous signal:

$$x(t) = cos(\frac{10\pi}{3}t) + sin(\frac{5\pi}{4}t)$$

The frequencies and periods of the two terms are, respectively,

$$\omega_1 = \frac{10\pi}{3},\ f_1 = \frac{5}{3},\ T_1 = \frac{3}{5}, \qquad \omega_2 = \frac{5\pi}{4},\ f_2 = \frac{5}{8},\ T_2 = \frac{8}{5}$$

The fundamental frequency f_0 is the GCD of $f_1 = 5/3$ and $f_2 = 5/8$:

$$f_0 = GCD(\frac{5}{3}, \frac{5}{8}) = GCD(\frac{40}{24}, \frac{15}{24}) = \frac{5}{24}$$

Alternatively, the period of the fundamental T_0 is the LCM of $T_1 = 3/5$ and $T_2 = 8/5$:

$$T_0 = LCM(\frac{3}{5}, \frac{8}{5}) = \frac{24}{5}$$

Now we get $\omega_0 = 2\pi f_0 = 2\pi/T_0 = 5\pi/12$ and the signal can be written as

$$x(t) = cos(8\frac{5\pi}{12}t) + sin(3\frac{5\pi}{12}t) = cos(8\omega_0 t) + sin(3\omega_0 t)$$

i.e., the two terms are the 3th and 8th harmonic of the fundamental frequency ω_0, respectively.

Example 2:

$$x(t) = sin(\frac{5\pi}{6}t) + cos(\frac{3\pi}{4}t) + sin(\frac{\pi}{3}t)$$

The frequencies and periods of the three terms are, respectively,

$$\omega_1 = \frac{5\pi}{6},\ f_1 = \frac{5}{12},\ T_1 = \frac{12}{5},\ \omega_2 = \frac{3\pi}{4},\ f_2 = \frac{3}{8},\ T_2 = \frac{8}{3},\ \omega_3 = \frac{\pi}{3},\ f_3 = \frac{1}{6},\ T_3 = 6$$

The fundamental frequency f_0 is the GCD of f_1, f_2 and f_3:

$$f_0 = GCD(\frac{5}{12}, \frac{3}{8}, \frac{1}{6}) = GCD(\frac{10}{24}, \frac{9}{24}, \frac{4}{24}) = \frac{1}{24}$$

Alternatively, the period of the fundamental T_0 is the LCM of T_1, T_2 and T_3:

$$T_0 = LCM(\frac{12}{5}, \frac{8}{3}, 6) = LCM(\frac{36}{15}, \frac{40}{15}, \frac{90}{15})\frac{5}{24}$$

The signal can be written as

$$x(t) = sin(\frac{10\pi}{12}t) + cos(\frac{9\pi}{12}t) + sin(\frac{4\pi}{12}t)$$

i.e., the fundamental frequency is $\omega_0 = \pi/12$, the fundamental period is $T_0 = 2\pi/\omega) = 24$, and the three terms are the 4th, 9th and 10th harmonic of ω_0, respectively.

Example 3: Find the fundamental frequency of the following continuous signal:

$$x(t) = cos(\frac{10}{3}t) + sin(\frac{5\pi}{4}t)$$

Here the angular frequencies of the two terms are, respectively,

$$\omega_1 = \frac{10}{3}, \quad \omega_2 = \frac{5\pi}{4}$$

The fundamental frequency ω_0 should be the GCD of ω_1 and ω_2:

$$\omega_0 = GCD(\frac{10}{3}, \frac{5\pi}{4})$$

which does not exist as π is an irrational number which cannot be expressed as a ratio of two integers, therefore the two frequencies can not be multiples of the same fundamental frequency. In other words, the signal as the sum of the two terms is not a periodic signal.

Fundamental Frequency of Discrete Signals

For a discrete complex exponential $x[n] = e^{j\omega_1 n}$ to be periodic with period N, it has to satisfy

$$e^{j\omega_1(n+N)} = e^{j\omega_1 n}, \quad \text{i.e.,} \quad e^{j\omega_1 N} = 1 = e^{j2\pi k}$$

that is, $\omega_1 N$ has to be a multiple of 2π:

$$\omega_1 N = 2\pi k, \quad \text{i.e.,} \quad \frac{\omega_1}{2\pi} = \frac{k}{N}$$

As k is an integer, $\omega_1/2\pi$ has to be a rational number (a ratio of two integers). In order for the period

$$N = k\frac{2\pi}{\omega_1}$$

to be the fundamental period, k has to be the smallest integer that makes N an integer, and the fundamental angular frequency is

$$\omega_0 = \frac{2\pi}{N} = \frac{\omega_1}{k}$$

The original signal can now be written as:

$$x[n] = e^{j\omega_1 n} = e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n}$$

Example 2: Show that a discrete signal

$$x[n] = e^{jm(2\pi/N)n}$$

has fundamental period

$$N_0 = N/gcd(N, m)$$

According to the discussion above, the fundamental period N_0 should satisfy

$$m\frac{2\pi}{N}N_0 = k2\pi, \quad \text{or} \quad N_0 = \frac{kN}{m} = \frac{N}{m/k}$$

We see that for N_0 to be an integer, $l \triangleq m/k$ has to divide N. But since $k = m/l$ is an integer, l also has to divide m. Moreover, since k needs to be the smallest integer satisfying the above equation, $l = m/k$ has to be the greatest common divisor of both N and m, i.e., $l = gcd(N, m)$, and the fundamental period can be written as

$$N_0 = \frac{N}{m/k} = \frac{N}{gcd(N, m)}$$

Example 2: Find the fundamental period of the following discrete signal:

$$x[n] = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$$

We first find the fundamental period for each of the two components.

- Assume the period of the first term is N_1, then it should satisfy

  $$e^{j(2\pi/3)(n+N_1)} = e^{j(2\pi/3)n} \cdot 1 = e^{j(2\pi/3)n} \cdot e^{jk2\pi} = e^{j(2\pi n/3+k2\pi)}$$

  where k is an integer. Equating the exponents, we have

  $$\frac{2\pi}{3}n + \frac{2\pi}{3}N_1 = \frac{2\pi}{3}n + k2\pi$$

  which can be solved to get $N_1 = 3k$. We find the smallest integer $k = 1$ for $N_1 = 3$ to be an integer, the fundamental period.

- Assume the period of the second term is N_2, then it should satisfy

  $$e^{j(3\pi/4)(n+N_2)} = e^{j(3\pi/4)n} \cdot 1 = e^{j(3\pi/4)n} \cdot e^{jk2\pi} = e^{j(3\pi n/4+k2\pi)}$$

  where k is an integer. Equating the exponents, we have

  $$\frac{3\pi}{4}n + \frac{3\pi}{4}N_2 = \frac{3\pi}{4}n + k2\pi$$

  which can be solved to get $N_2 = 8k/3$. We find the smallest integer $k = 3$ for $N_2 = 8$ to be an integer, the fundamental period. Now the second term can be written as

  $$e^{j\frac{3\pi}{4}n} = e^{j2\frac{3\pi}{8}n}$$

Given the fundamental periods $N_1 = 3$ and $N_2 = 8$ of the two terms, the fundamental period N_0 of their sum is easily found to be their least common multiple

$$N_0 = lcm(3, 8) = 24$$

and the fundamental frequency is

$$\omega_0 = \frac{2\pi}{24} = \frac{\pi}{12}$$

Now the original signal can be written as

$$x[n] = e^{8\frac{2\pi}{24}n} + e^{9\frac{2\pi}{24}n}$$

i.e., the two terms are the 8th and 9th harmonic of the fundamental frequency $\omega_0 = \pi/12$.

Similar to the continuous case, to find the fundamental frequecy of a signal containing multiple terms all expressed as a fraction multiplied by π, we can rewrite these fractions in terms of the least common multiple of all the denominators.

Example 3:

$$x[n] = sin(\frac{5\pi}{6}n) + cos(\frac{3\pi}{4}n) + sin(\frac{\pi}{3}n)$$

The least common multiple of the denominators is 12, therefore

$$x[n] = sin(\frac{10\pi}{12}n) + cos(\frac{9\pi}{12}n) + sin(\frac{4\pi}{12}n)$$

i.e., the fundamental frequency is $\omega_0 = \pi/12$, the fundamental period is $T = 2\pi/\omega_0 = 24$ and the three terms are the 4th, 9th and 10th harmonic of ω_0, respectively.