E101 Handout 2: Fourier Series Expansion

Fourier Series Expansion

Consider a signal $x(t)$ composed of a set of sinusoids of different frequencies and magnitudes:

$$x(t) = cos(0t) + cos(3t) - \frac{1}{3}cos(3t) + \frac{1}{5}cos(5t) - \frac{1}{7}cos(7t)$$

The angular frequencies $\omega$ of the components (excluding the DC component) are, respectively, 1, 3, 5 and 7. Their greatest common divisor $\omega_0 = 1$ is the fundamental frequency of the signal $x(t)$. On the other hand, the periods $T = 2\pi/\omega$ of the components (excluding DC) are, respectively, $2\pi, 2\pi/3, 2\pi/5$ and $2\pi/7$. Their least common multiple $T = 2\pi/\omega_0$ is the period of $x(t)$.

According to Euler’s formula:

$$\left\{ \begin{array}{l}
cos(\omega_0 t) = (e^{j\omega_0 t} + e^{-j\omega_0 t})/2 \\
sin(\omega_0 t) = (e^{j\omega_0 t} - e^{-j\omega_0 t})/2j
\end{array} \right.$$

the signal can also be expressed as a linear combination of complex exponentials:

$$x(t) = e^{0t} + \frac{1}{2}[(e^{j\omega_0 t} + e^{-j\omega_0 t}) - \frac{1}{3}(e^{j3\omega_0 t} + e^{-j3\omega_0 t}) + \frac{1}{5}(e^{j5\omega_0 t} + e^{-j5\omega_0 t}) - \frac{1}{7}(e^{j7\omega_0 t} + e^{-j7\omega_0 t})] = \sum_{k=-7}^{7} X[k]e^{jk\omega_0 t}$$

where $\omega_0 = 1$ is the fundamental frequency and the coefficients are

In general, a periodic signal \( x_T(t) = x_T(t + T) \) can be expressed as a linear combination of infinite complex exponentials called Fourier series:

\[
x_T(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\omega_0 t}
\]

The coefficients \( X[k] \) (\( a_k \) in OWN) can be found in two steps:

1. multiply both sides by \( e^{-j\omega_0 t} / T \):

\[
\frac{1}{T} x_T(t) e^{-j\omega_0 t} = \frac{1}{T} \sum_{k=-\infty}^{\infty} X[k] e^{j\omega_0 t} e^{-j\omega_0 t} = \frac{1}{T} \sum_{k=-\infty}^{\infty} X[k] e^{j(k-n)\omega_0 t}
\]

2. integrate both sides with respect to \( t \) over one period \( T = 2\pi / \omega_0 \):

\[
\frac{1}{T} \int_T x_T(t) e^{-j\omega_0 t} dt = \frac{1}{T} \sum_{k=-\infty}^{\infty} X[k] \int_T e^{j(k-n)\omega_0 t} dt = X[n]
\]

where

\[
\frac{1}{T} \int_T e^{j(k-n)\omega_0 t} dt = \frac{1}{T} \int_T e^{j(k-n)2\pi t/T} dt = \delta[k-n] = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}
\]

as the integral over interval \( T \) of \( e^{j(k-n)\omega_0 t} \) (composed of sine and cosine functions of period \( T \)) is zero unless \( k = n \) in which case the integral is \( T \). Therefore we get Fourier coefficient for the \( k \)th term:

\[
X[k] = \frac{1}{T} \int_T x_T(t) e^{-j\omega_0 t} dt = \frac{1}{T} \int_T x_T(t) e^{-j2\pi k \omega_0 t} dt
\]
These coefficients $X[k]$ are also called *spectral coefficients* and can be represented as a set of weighted impulses over the continuous frequency axis with an interval $\omega_0 = 2\pi f_0$ between two consecutive impulses:

$$X(\omega) = \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - k\omega_0), \quad \text{or} \quad X(f) = \sum_{k=-\infty}^{\infty} X[k] \delta(f - k f_0)$$

This continuous function of frequency is called the *spectrum* of the signal.
Physical Interpretation of Fourier Expansion

What is the physical meaning of a spike in the spectrum $X(\omega)$ of a signal $x(t)$? Does a Fourier coefficient such as $X[-1]$ represent a frequency component of “negative frequency” $-\omega_0$? To answer these questions, we rewrite the Fourier expansion as

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = X[0] + \sum_{k=1}^{\infty}[X[k]e^{jk\omega_0 t} + X[-k]e^{-jk\omega_0 t}]$$

$$= X[0] + \sum_{k=1}^{\infty}[X[k](\cos(k\omega_0 t) + j \sin(k\omega_0 t)) + X[-k](\cos(k\omega_0 t) - j \sin(k\omega_0 t))]$$

$$= X[0] + \sum_{k=1}^{\infty}[(X[k] + X[-k])\cos(k\omega_0 t) + j(X[k] - X[-k])\sin(k\omega_0 t)]$$

$$= X[0] + \sum_{k=1}^{\infty}[A[k]\cos(k\omega_0 t) + B[k]\sin(k\omega_0 t)]$$

where

$$X[0] = \frac{1}{T} \int_{-T}^{T} x(t)e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-T}^{T} x(t) dt$$

is the average or DC component of the signal $x(t)$, and $A[k]$ and $B[k]$ are two complex coefficients:

$$A[k] \triangleq X[k] + X[-k] = \frac{1}{T} \int_{-T}^{T} x(t)(e^{-jk\omega_0 t} + e^{jk\omega_0 t}) dt = \frac{2}{T} \int_{-T}^{T} x(t) \cos(k\omega_0 t) dt$$

$$B[k] \triangleq j(X[k] - X[-k]) = \frac{j}{T} \int_{-T}^{T} x(t)(e^{-jk\omega_0 t} - e^{jk\omega_0 t}) dt = \frac{2}{T} \int_{-T}^{T} x(t) \sin(k\omega_0 t) dt$$

This alternative form of the Fourier expansion no longer contains any terms corresponding to “negative frequencies” $\omega = k\omega_0 < 0$. The key step in this conversion is to combine the two corresponding coefficients $X[k]$ and $X[-k]$ for complex exponentials to get two different coefficients $A[k]$ and $B[k]$ for sine and cosine functions.

In general, the signal $x(t)$ is complex and so are the coefficients $X[k]$ (or $A[k]$ and $B[k]$). The above expression for $x(t)$ is a complex equation containing both real and imaginary parts.

In particular, if the signal is real $x^*(t) = x(t)$, the complex conjugate of its Fourier coefficient is

$$X^*[k] = \frac{1}{T} \int_{-T}^{T} x(t)e^{-jk\omega_0 t} dt^* = \frac{1}{T} \int_{-T}^{T} x(t)e^{jk\omega_0 t} dt = X[-k]$$

and the coefficients $A[k]$ and $B[k]$ can be written as

$$\begin{cases} A[k] = X[k] + X[-k] = X[k] + X^*[k] = 2X_r[k] = 2|X[k]|\cos(\angle X[k]) \\ B[k] = X[k] - X[-k] = X[k] - X^*[k] = 2jX_i[k] = 2j|X[k]|\sin(\angle X[k]) \end{cases}$$

The Fourier expression of the signal becomes

$$x(t) = X[0] + \sum_{k=1}^{\infty} 2|X[k]| \left[ \cos(\angle X[k])\cos(k\omega_0 t) - \sin(\angle X[k])\sin(k\omega_0 t) \right]$$

We see that any real periodic signal $x(t)$ can be expressed as the weighted sum of its DC component $X[0]$, its fundamental frequency component $\cos(k\omega_0 t)$ of magnitude $2|X[1]|$ and phase shift $\angle X[1]$, and the linear combination of an infinite number of sinusoidal harmonics $\cos(k\omega_0 t)$ ($k > 1$) with magnitude $2|X[k]|$ and phase shift $\angle X[k]$. 

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Properties of Fourier Expansion

A continuous and periodic signal $x(t) = x(t + T)$ and its Fourier expansion can be described as:

$$x(t) \iff X[n]$$

or more specifically

$$x(t) = \mathcal{F}^{-1}[X[n]] = \sum_{n=-\infty}^{\infty} X[n] e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} X[n] e^{j2\pi f_0 t}$$

$$X[n] = \mathcal{F}[x(t)] = \frac{1}{T} \int_{T} x(t) e^{-jn\omega_0 t} = \frac{1}{T} \int_{T} x(t) e^{-j2\pi n f_0 t}$$

where $\mathcal{F}$ represents Fourier expansion, and $\omega_0 = 2\pi f_0 = 2\pi / T$.

Some of the important properties of Fourier expansion are summarized below. We always assume

$$\mathcal{F}[x(t)] = X[k], \quad \text{i.e.,} \quad \mathcal{F}^{-1}[X[k]] = x(t)$$

- **Linearity**

  The Fourier expansion can be considered as a linear operation applied to the signal:

  $$\mathcal{F}[x(t)] = \frac{1}{T} \int_{T} x(t) e^{-jk\omega_0 t} dt$$

  $$\mathcal{F}[ax(t) + by(t)] = \frac{1}{T} \int_{T} [ax(t) + by(t)] e^{-jk\omega_0 t} dt$$

  $$= a \frac{1}{T} \int_{T} x(t) e^{-jk\omega_0 t} dt + b \frac{1}{T} \int_{T} y(t) e^{-jk\omega_0 t} dt$$

  $$= aX[k] + bY[k]$$
• Time Reversal

The time reversed version of a signal $x(t)$ is $x(-t)$, and its Fourier coefficient can be found to be:

$$\mathcal{F}[x(-t)] = \frac{1}{T} \int_{0}^{T} x(-t)e^{-jk\omega_0 t} dt$$

We let $t' = -t$ and get

$$\mathcal{F}[x(-t)] = \frac{1}{T} \int_{0}^{T} x(t')e^{jk\omega_0 t'} dt'$$

We further let $k' = -k$ and get

$$\mathcal{F}[x(-t)] = \frac{1}{T} \int_{0}^{T} x(t')e^{-jk'\omega_0 t'} dt' = X[k'] = X[-k]$$

• Time and Frequency Scaling

When the time axis is scaled by a factor $a > 0$, then a signal $x(t)$ becomes $x(at)$. We see that

if $x(t) = x(t + T)$, then $x(at) = x(at + T) = x(a(t + T/a))$

i.e., the period of a time scaled signal $x(at)$ becomes $T/a$. If $a > 1$, $x'(t) = x(at)$ is compressed and has a smaller period, while if $0 < a < 1$, $x(at)$ is stretched and has a larger period.

Example: $x(t) = \cos(\omega_0 t)$ is a periodic signal with period $T = 1/f = 2\pi/\omega_0$:

$$x(t + T) = \cos(\omega_0 (t + T)) = \cos(\omega_0 t + \omega_0 T)$$

$$= \cos(\omega_0 t + 2\pi) = \cos(\omega_0 t) = x(t)$$

If the time axis is scaled: $y(t) = x(at) = \cos(\omega_0 at)$, the period becomes $T/a$:

$$y(t + \frac{T}{a}) = \cos(\omega_0 a(t + \frac{T}{a})) = \cos(\omega_0 at + \omega_0 T)$$

$$= \cos(\omega_0 at + 2\pi) = \cos(\omega_0 at) = y(t)$$

In general, if $y(t) = x(at)$ is the scaled version of $x(t)$ with period $T$, then its Fourier coefficient is

$$Y[k] = \frac{1}{T} \int_{0}^{T} x(at)e^{-jk\omega_0 t} dt$$

We let $t' = at$, i.e., $t = t'/a$ and get

$$Y[k] = \frac{1}{T} \int_{0}^{aT} x(t')e^{-jk\omega_0 t'/a} \frac{dt'}{a} = X[k/a]$$

Or, in the time domain, the Fourier series of a time scaled signal is

$$x(at) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} X[k]e^{j(k\omega_0)t}$$

We see that the same coefficient $X[k]$ is now the weight for a different complex exponential with frequency $k\omega_0$. If $a > 1$, the impulse in the spectrum representing $X[k]$ is located at $\omega = k\omega_0$ on the frequency axis, $a$ times farther away from the origin than its original location $\omega = \omega_0$ corresponding to the unscaled signal $x(t)$. In other words, when $a > 1$, the signal $x(at)$ is compressed while its spectrum is stretched, and vice versa.

Example: The spectrum of a sinusoidal signal

$$x(t) = \cos(\omega_0 t) = (e^{j\omega_0 t} + e^{-j\omega_0 t})/2$$

is composed of two impulses located at $\omega = \omega_0$ and $\omega = -\omega_0$, respectively. When the signal is scaled:

$$x(at) = \cos(\omega_0 at) = (e^{j\omega_0 at} + e^{-j\omega_0 at})/2$$

the two impulses in the spectrum are located at $\omega = a\omega_0$ and $\omega = -a\omega_0$ respectively.
• Time Shift
When signal $x(t)$ is shifted in time by $t_0$, it becomes $x'(t) = x(t - t_0)$. Note that $x(t)$ is shifted forward to the right if $t_0 > 0$, or backward to the left if $t_0 < 0$. The corresponding Fourier expansion becomes:

$$X'[k] = \frac{1}{T} \int_T x(t - t_0)e^{-j\omega_0 t} dt$$

Letting $t' = t - t_0$, we have $t = t' + t_0$, and

$$X'[k] = \frac{1}{T} \int_T x(t')e^{-j\omega_0 (t' + t_0)} dt$$

$$= e^{-j\omega_0 t_0} \frac{1}{T} \int_T x(t')e^{-j\omega_0 t'} dt' = e^{-j\omega_0 t_0} X[k]$$

We see that when the signal is shifted in time by $t_0$, its Fourier coefficient is multiplied by a phase factor $e^{-j\omega_0 t_0}$, so that the corresponding kth complex exponential $e^{j\omega_0 t} = \cos(k\omega_0 t) + j \sin(k\omega_0 t)$ is phase-shifted by $-k\omega_0 t_0$ to become

$$e^{j\omega_0 t - j\omega_0 t_0} = e^{j\omega_0 (t - t_0)} = \cos(k\omega_0 (t - t_0)) + j \sin(k\omega_0 (t - t_0))$$

while its magnitude remains the same.

![Graph showing Fourier transform of a shifted signal](image)

• Frequency Shift
When the spectrum $X[k]$ of a signal $x(t)$ is right shifted by $l$ to become $Y[k] = X[k - l]$, the corresponding signal $y(t)$ in time domain is:

$$y(t) = \sum_{k=-\infty}^{\infty} Y[k]e^{j(k-l)\omega_0 t} = \sum_{k=-\infty}^{\infty} X[k-l]e^{jk\omega_0 t}$$

Letting $k' = k - l$, we have $k = k' + l$ and

$$y(t) = \sum_{k'=-\infty}^{\infty} X[k']e^{j(k'+l)\omega_0 t} = e^{j\omega_0 lt} \sum_{k'=-\infty}^{\infty} X[k']e^{jk'\omega_0 t}$$

$$= e^{j\omega_0 lt} x(t) = [\cos(l\omega_0 t) + j \sin(l\omega_0 t)]x(t)$$

i.e., the signal $x(t)$ is modulated by a sinusoid of frequency $l\omega_0$. Note that if $X[k]$ is left shifted by $l$ to become $X[k + l]$, the corresponding signal is also modulated by the same frequency

$$y(t) = e^{-j\omega_0 lt} x(t) = [\cos(l\omega_0 t) - j \sin(l\omega_0 t)]x(t)$$
• Convolution Theorem - Time Domain
If both signals \( x(t) = x(t + T) \) and \( y(t) = y(t + T) \) are periodic, then their periodic convolution is defined as:

\[
x(t) * y(t) \triangleq \frac{1}{T} \int_{T} x(\tau)y(t - \tau) d\tau
\]

The Fourier expansion of this periodic convolution is

\[
\mathcal{F}[x(t) * y(t)] = \frac{1}{T} \int_{T} \left[ \int_{T} x(\tau) y(t - \tau) d\tau \right] e^{-j\omega t} dt
\]

Letting \( t' = t - \tau \), we get \( t = t' + \tau \) and

\[
\mathcal{F}[x(t) * y(t)] = \frac{1}{T} \int_{T} x(\tau) e^{-j\omega \tau} \frac{1}{T} \left[ \int_{T} y(t') e^{-j\omega \tau} dt' \right] d\tau
\]

\[
= \frac{1}{T} \int_{T} x(\tau) e^{-j\omega \tau} Y[k] d\tau
\]

\[
= X[k] Y[k]
\]

• Convolution Theorem - Frequency Domain
The Fourier expansion of the product of two signals \( x(t)y(t) \) is

\[
\mathcal{F}[x(t)y(t)] = \frac{1}{T} \int_{T} [x(t)y(t)] e^{-j\omega t} dt
\]

But as

\[
x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\omega kt}, \quad y(t) = \sum_{k=-\infty}^{\infty} Y[k] e^{j\omega kt}
\]

we have

\[
\mathcal{F}[x(t)y(t)] = \frac{1}{T} \int_{T} \left[ \sum_{m=-\infty}^{\infty} X[m] e^{j\omega mt} \sum_{n=-\infty}^{\infty} X[n] e^{j\omega nt} \right] e^{-j\omega t} dt
\]
\[
= \sum_{m=-\infty}^{\infty} X[m] \sum_{n=-\infty}^{\infty} Y[n] \frac{1}{T} \int_T e^{j(m+n-k)\omega_0 t} dt
\]

\[
= \sum_{m=-\infty}^{\infty} X[m] Y[n] \delta[m + n - k]
\]

\[
= \sum_{m=-\infty}^{\infty} X[m] Y[k - m] \triangleq X[k] * Y[k]
\]

where we used the relation

\[
\frac{1}{T} \int_T e^{j(m+n-k)\omega_0 t} dt = \delta[m + n - k] = \begin{cases} 
1 & \text{if } m + n - k = 0 \\
0 & \text{else}
\end{cases}
\]

Comparing the convolution theorems in both time and frequency domains, its duality can be observed:

\[
\begin{align*}
\mathcal{F}[x(t)y(t)] &= X[k] * Y[k] \\
\mathcal{F}[x(t) * y(t)] &= X[k]Y[k]
\end{align*}
\]
• **Complex Conjugation**

In general, a signal \( x(t) \) considered in Fourier transform is a complex function containing both real and imaginary parts:

\[
x(t) = x_r(t) + jx_i(t)
\]

and its complex conjugate is

\[
x^*(t) = x_r(t) - jx_i(t)
\]

To find the Fourier coefficient of \( x^*(t) \), we take complex conjugation on both sides of the Fourier expansion of \( x(t) \):

\[
x^*(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} = \lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} X^*[k] e^{-jk\omega_0 t}
\]

and replace \( k \) by \(-k\)

\[
x^*(t) = \sum_{k=-\infty}^{\infty} X^*[-k] e^{jk\omega_0 t}
\]

i.e.,

\[\text{if } \mathcal{F}[x(t)] = X[k] \text{ then } \mathcal{F}[x^*(t)] = X^*[-k]\]

• **Parseval’s Relation**

In the equation for the theorem of frequency convolution

\[
\mathcal{F}[x(t)y(t)] = \frac{1}{T} \int_{T} x(t)y(t) e^{-j\omega_0 t} dt = \sum_{m=-\infty}^{\infty} X[m] Y[k-m]
\]

if we assume (a) \( k = 0 \) and (b) \( y(t) = x^*(t) \), and therefore \( Y[k-m] = X^*[m-k] \) (property of complex conjugation discussed above), we get the Parseval’s theorem

\[
\frac{1}{T} \int_{T} |x(t)|^2 dt = \sum_{m=-\infty}^{\infty} |X[m]|^2
\]

The left hand side of the equation is the average power (energy per unit time) in one period of the signal in time domain, while the right hand side is the sum of the power contained in each frequency component (the kth harmonic) of the signal:

\[
\frac{1}{T} \int_{T} |X[m] e^{jm\omega_0 t}|^2 dt = \frac{1}{T} \int_{T} |X[m]|^2 dt = |X[m]|^2
\]

The Parseval’s equation indicates that the total amount of energy (or information) contained in the signal is the same in either time or frequency domain, i.e., the signal can be expressed in either time or frequency domain without any energy gained or lost.
Fourier Series as Input to LTI

An LTI system’s response to a complex exponential input $x(t) = e^{j\omega_0 t}$ can be easily found by its eigenvalue equation:

$$y(t) = \mathcal{O}[e^{j\omega_0 t}] = H(s)e^{j\omega_0 t}$$

where the eigenvalue, the transfer function $H(s)$, can be obtained from its impulse response function

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

When $s = j\omega$, the transfer function becomes the frequency response function:

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

This approach can be extended from complex exponential input to an arbitrary periodic input $x(t)$ with period $T = 2\pi/\omega_0$, which can be expressed as a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\omega_0 t}, \text{ where } X[k] = \frac{1}{T} \int_{T} x(t)e^{-j\omega_0 t} dt$$

The output of the system is

$$y(t) = \mathcal{O}[x(t)] = \mathcal{O}\left[ \sum_{k=-\infty}^{\infty} X[k]e^{j\omega_0 t} \right] = \sum_{k=-\infty}^{\infty} X[k]\mathcal{O}[e^{j\omega_0 t}]$$

$$= \sum_{k=-\infty}^{\infty} X[k]H(jk\omega_0)e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} X[k]H[k]e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} Y[k]e^{j\omega_0 t}$$

where

$$H[k] \triangleq H(jk\omega_0) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega_0 \tau}d\tau$$

is the eigenvalue of the system corresponding to the eigenfunction, the complex exponential input $x(t) = e^{j\omega_0 t}$, and

$$Y[k] \triangleq X[k]H[k] = \frac{1}{T} \int_{T} y(t)e^{-j\omega_0 t} dt$$

happens to be the Fourier coefficient of response $y(t)$, which is also periodic with the same period $T$.

The result above indicates that an LTI system’s response to a weighted sum of complex exponentials $e^{j\omega_0 t}$ is the weighted sum (with the same weights $X[k]$) of its responses to individual exponentials. Moreover, the response can also be considered as the sum of the complex exponentials weighted by $Y[k] = H[k]X[k]$.

If both the input and output are represented by their Fourier series coefficients:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j\omega_0 t}, \quad y(t) = \sum_{k=-\infty}^{\infty} Y[k]e^{j\omega_0 t}$$

the convolution describing the LTI in time domain

$$y(t) = h(t) \ast x(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

becomes multiplication in frequency domain:

$$Y[k] = H[k]X[k]$$
This simplification is one of the reasons why we are interested in complex exponentials (or equivalently sinusoidal functions) and Fourier transform of different forms (including Laplace and Z transforms) while studying LTI systems. However, analyzing the signals and systems in transform domain (Fourier, Laplace, Z, etc.) has many more important advantages and applications, some of them will be discussed later.

Consider a sinusoidal signal

\[ x(t) = \cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t}) \]

as the input to an LTI system. The output is

\[ y(t) = \frac{1}{2}[H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}] \]

For a real system \( h(t) = h^*(t) \), we have (complex conjugate property):

\[ H(-j\omega) = H^*(j\omega) = \left( |H(j\omega)|e^{j\angle H(j\omega)} \right)^* = |H(j\omega)|e^{-j\angle H(j\omega)} \]

and the output becomes:

\[ y(t) = |H(j\omega)|\frac{1}{2}[e^{j\angle H(j\omega)}e^{j\omega t} + e^{-\angle H(j\omega)}e^{-j\omega t}] = |H(j\omega)|\cos(\omega t + \angle H(j\omega)) \]

i.e., the response of an LTI to a sinusoidal is also a sinusoidal of the same frequency, scaled by \( H(j\omega) \) and phase shifted by \( \angle H(j\omega) \).
Fourier Expansion of Typical Signals

- Complex exponential
  
  If the signal is a complex exponential $x(t) = e^{j\omega_0 t} = e^{j2\pi f_0 t}$ with period $T = 2\pi / \omega_0 = 1 / f_0$, its Fourier coefficient can be found to be

  $$X[k] = \mathcal{F}[e^{j\omega_0 t}] = \frac{1}{T} \int_T e^{j\omega_0 t} e^{-j\omega_0 k t} dt$$

  $$= \frac{1}{T} \int_T e^{j(1-k)\omega_0 t} dt = \delta[k - 1] = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

  Similarly, we have

  $$X[k] = \mathcal{F}[e^{-j\omega_0 t}] = \frac{1}{T} \int_T e^{-j\omega_0 t} e^{-j\omega_0 k t} dt = \frac{1}{T} \int_T e^{j(1+k)\omega_0 t} dt = \delta[k + 1]$$

  In general, the Fourier coefficients for $x(t) = e^{\pm jn\omega_0 t}$ is

  $$X[k] = \mathcal{F}[e^{\pm jn\omega_0 t}] = \frac{1}{T} \int_T e^{\pm jn\omega_0 t} e^{-j\omega_0 k t} dt = \frac{1}{T} \int_T e^{-j(k \mp n)\omega_0 t} dt = \delta[k \pm n]$$

- Constant

  In particular, when $n = 0$, the above complex exponential function $x(t) = e^{jn\omega_0 t} = e^0 = 1$ becomes a constant, and its spectrum is

  $$X[k] = \mathcal{F}[1] = \delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

  That is, the signal only has the DC component $X[0]$. 


• Sinusoids

Based on the results above and the linearity of the Fourier series, the spectrum of a sinusoidal signal
\( x(t) = \sin(\omega_0 t) = \frac{1}{2j} \left[ e^{j\omega_0 t} - e^{-j\omega_0 t} \right] \) with period \( T = 2\pi/\omega_0 \) is

\[
X[k] = \mathcal{F}[x(t)] = \frac{1}{2j} \left\{ \mathcal{F}[e^{j\omega_0 t}] - \mathcal{F}[e^{-j\omega_0 t}] \right\} = \frac{1}{2j} [\delta[k-1] - \delta[k+1]]
\]

We see that all terms in the Fourier series are zero except two terms, \( X[1] = 1/2j \) and \( X[-1] = -1/2j \), and the Fourier series becomes

\[
x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} = \frac{1}{2j} [e^{j\omega_0 t} - e^{-j\omega_0 t}] = \sin(\omega_0 t)
\]

Similarly, if \( x(t) = \cos(\omega_0 t) \), we have

\[
X[k] = \frac{1}{2} [\delta[k-1] + \delta[k+1]]
\]

and

\[
x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{jk\omega_0 t} = \frac{1}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}] = \cos(\omega_0 t)
\]

**Example 1**

\[
x_1(t) = \cos(\omega_1 t) = \cos(4\pi t) = \frac{1}{2} (e^{4\pi i t} + e^{-4\pi i t})
\]

The fundamental frequency of the signal is \( \omega_1 = 4\pi \), or the period is \( T_1 = 2\pi/\omega_1 = 0.5 \), and

\[
X_1[k] = \frac{1}{2} [\delta[k+1] + \delta[k-1]]
\]

The distance from each of the two unit samples to the origin is \( \omega_1 = 4\pi \).

**Example 2**

\[
x_2(t) = \cos(\omega_2 t) = \cos(5\pi t) = \frac{1}{2} (e^{5\pi i t} + e^{-5\pi i t})
\]

The fundamental frequency of the signal is \( \omega_2 = 5\pi \), or the period is \( T_2 = 2\pi/\omega_2 = 0.4 \), and

\[
X_2[k] = \frac{1}{2} [\delta[k+1] + \delta[k-1]]
\]

The distance from each of the two unit samples to the origin is \( \omega_2 = 5\pi \).

**Example 3**

Now consider the combination of the two signals above:

\[
x_3(t) = x_1(t) + x_2(t) = \cos(4\pi t) + \cos(5\pi t)
\]

The fundamental frequency of \( x_3(t) = x_1(t) + x_2(t) \) is the greatest common divisor of their individual frequencies \( \omega_1 \) and \( \omega_2 \):

\[
\omega_3 = \gcd(\omega_1, \omega_2) = \gcd(4\pi, 5\pi) = \pi
\]

Or, equivalently, the period of \( x_3(t) \) is the least common multiple of their individual periods \( T_1 \) and \( T_2 \):

\[
T_3 = \text{lcm}(T_1, T_2) = \text{lcm}(0.5, 0.4) = 2
\]
The signal can be rewritten as
\[ x_3(t) = \cos(4\omega_0 t) + \cos(5\omega_0 t) = \frac{1}{2} \left[ e^{j4\pi t} + e^{-j4\pi t} + e^{j5\pi t} + e^{-j5\pi t} \right] \]
and the Fourier coefficients are
\[ X[k] = \frac{1}{2} \left[ \delta[k + 4] + \delta[k - 4] + \delta[k + 5] + \delta[k - 5] \right] \]
Note that the interval between two neighboring frequencies along the frequency axis is \( \omega_3 = \pi \).

\[ \bullet \text{Square wave (even)} \]
Let \( x(t) \) be an even square wave:
\[ x(t) = \begin{cases} 
1 & |t| < \tau/2 \\
0 & \tau/2 < |t| < T/2 
\end{cases} \]
This is a train of rectangular impulses with width \( \tau \) and period \( T \). The Fourier coefficients of this function is
\[
X[k] = \frac{1}{T} \int_T x(t) e^{-j\omega_0 t} dt = \frac{1}{T} \int_{-\tau/2}^{\tau/2} e^{-j\omega_0 t} dt
\]
\[
= \frac{2}{T\omega_0} \left[ \frac{1}{2j} \left( e^{j\omega_0 \tau/2} - e^{-j\omega_0 \tau/2} \right) \right] = \frac{2}{T\omega_0} \sin(\omega_0 \tau/2)
\]
\[
= \frac{\tau \sin(\omega_0 \tau/2)}{k\omega_0 \tau/2} = \frac{\tau}{T} \frac{\sin(\pi k\tau/T)}{\pi k\tau/T} = \frac{\tau}{T} \text{sinc}(k\tau/T)
\]
where the sinc function is defined as
\[ \text{sinc}(\theta) \triangleq \frac{\sin(\pi \theta)}{\pi \theta}, \quad \text{and} \quad \lim_{\theta \to 0} \text{sinc}(\theta) = 1 \]
When \( k = 0 \), we get the DC component:
\[ X[0] = \frac{T}{T} \text{sinc}(k\tau/T)|_{k=0} = \frac{\tau}{T} \]
Note that \( x[k] = 0 \) iff its numerator \( \sin(k\pi\tau/T) = 0 \), i.e., \( k\pi\tau/T = \pm n\pi \) or \( k = \pm nT/\tau \). In other words, when the ratio \( T/\tau \) is an integer, then \( X[k] = 0 \) if \( k \) is a multiple of \( T/\tau \). But if \( T/\tau \) is not an integer, \( X[k] \neq 0 \) for any \( k \).
We further consider two special cases:

(1) If $\tau = T$, the square wave becomes constant 1, and its Fourier coefficient becomes

$$X[k] = \frac{\sin(k\pi)}{k\pi} = \text{sinc}[k] = \delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

i.e., the function only has the DC component:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi ft} = e^0 = 1$$

(2) If $\tau = T/2$, we have

$$X[k] = \frac{\sin(\pi k/2)}{\pi k} \quad (k = 0, \pm 1, \pm 2, \cdots)$$

It can be seen that

- $X[0] = 1/2$.
- $X[n]$ is an even function:

$$X[-n] = \frac{\sin(-\pi n/2)}{-\pi n} = \frac{\sin(\pi n/2)}{\pi n} = X[n]$$

- When $n = \pm 2k$ is an even number, $X[n] = 0$.
- When $n = \pm (2k - 1)$ is an odd number,

$$X[n] = \frac{(-1)^{k-1}}{(2k-1)\pi} \quad (k = 1, 2, 3, \cdots)$$

i.e.,


$$X[5] = X[-5] = 1/5\pi; \quad X[7] = X[-7] = -1/7\pi; \cdots$$

The square wave can now be represented as a Fourier series with all even terms dropped and every two corresponding odd terms $X[-k]$ and $X[k]$ combined:

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\omega_0 t}$$

$$= X[0] + \sum_{k=1}^{\infty} X[2k-1] \{ e^{j(2k-1)\omega_0 t} + e^{-j(2k-1)\omega_0 t} \}$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \cos((2k-1)\omega_0 t)$$

$$= \frac{1}{2} + \frac{2}{\pi} \left[ \frac{\cos(\omega_0 t)}{1} - \frac{\cos(3\omega_0 t)}{3} + \frac{\cos(5\omega_0 t)}{5} - \frac{\cos(7\omega_0 t)}{7} + \cdots \right]$$

- **Square wave (odd)**

Let $x(t)$ be an odd square wave:

$$x(t) = \begin{cases} 1 & 0 < t < \tau \\ 0 & \tau < t < T \end{cases}$$
The Fourier coefficients of this function is

\[
X[k] = \frac{1}{T} \int_{\tau} x(t)e^{-j\omega_0 t} dt = \frac{1}{T} \int_{0}^{\tau} e^{-j\omega_0 t} dt
\]

\[
= \frac{1}{Tj\omega_0} \left[ 1 - e^{-j\omega_0 \tau} \right] = \frac{2e^{-j\omega_0 \tau/2}}{Tk\omega_0} \frac{1}{2j} \left[ e^{j\omega_0 \tau/2} - e^{-j\omega_0 \tau/2} \right]
\]

\[
= \frac{2e^{-j\omega_0 \tau/2}}{Tk\omega_0} \sin(k\omega_0 \tau/2) = \frac{\tau}{T} \frac{\sin(\pi k\tau/T)}{\pi k\tau/T} e^{-j\omega_0 \tau/2}
\]

\[
= \frac{\tau}{T} \text{sinc}(k\tau/T)e^{-j\omega_0 \tau/2}
\]

Comparing this result with that of the even square wave, the time shift theorem is confirmed, i.e., when the signal is shift in time by \(\tau/2\), its Fourier coefficient is multiplied by a phase factor \(e^{-j\omega_0 \tau/2}\).

Again, when \(\tau = T/2\), we have

\[
X[k] = \frac{1}{j2\pi k} [1 - e^{j\pi k}] \quad (k = 0, \pm 1, \pm 2, \cdots)
\]

It can be seen that

- \(X[0] = 1/2\).
- When \(n = \pm 2k\) is an even number \((k = 1, 2, \cdots), X[n] = 0\)
- When \(n = \pm (2k - 1)\) is an odd number \((k = 1, 2, \cdots), X[n] = 1/j\pi n\)

The square wave can now be represented as a Fourier series with all even terms dropped and every two corresponding odd terms \(X[-k]\) and \(X[k]\) combined:

\[
x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j\omega_0 t}
\]

\[
= X[0] + \sum_{k=1}^{\infty} \frac{1}{j\pi (2k - 1)} e^{j(2k-1)\omega_0 t} + \frac{1}{-j\pi (2k - 1)} e^{-j(2k-1)\omega_0 t}
\]

\[
= \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k - 1} \sin((2k - 1)\omega_0 t)
\]

\[
= \frac{1}{2} + \frac{2}{\pi} \left[ \frac{\sin(\omega_0 t)}{1} + \frac{\sin(3\omega_0 t)}{3} + \frac{\sin(5\omega_0 t)}{5} + \frac{\sin(7\omega_0 t)}{7} + \cdots \right]
\]