E101 Handout 4: Discrete Fourier Transform

Fourier Transform of Discrete Periodic Signal

If a given signal \( x(t) = x(t+T) \) is periodic (temporal period \( T \)), its spectrum \( X[n] \) is discrete (Fourier series with interval \( \omega_0 = 2\pi/T \) or \( f_0 = 1/T \)). If a given signal \( x[m] \) is discrete (with temporal interval \( t_0 \) between two samples), its spectrum \( X_F(f) \) is periodic (frequency period \( \Omega = 2\pi/t_0 \) or \( F = 1/t_0 \)). Obviously when the signal is both periodic with time period \( T \) and discrete with time interval \( t_0 \), then its spectrum should be both discrete with frequency interval \( \omega_0 = 2\pi/T \) and periodic with frequency period \( \Omega = 2\pi/t_0 \). If there are \( N \) samples in a time period \( T \):

\[
N \triangleq \frac{T}{t_0}
\]

then there are also \( N \) frequency components in a frequency period \( \Omega \):

\[
\frac{\Omega}{\omega_0} = \frac{F}{f_0} = \frac{1}{t_0} = \frac{T}{t_0} = N
\]

Also we have

\[
f_0 t_0 = \frac{t_0}{T} = \frac{1}{N}, \quad TF = \frac{T}{t_0} = N
\]

To derive the Fourier transform of a periodic and discrete signal, we first discretize a periodic signal \( x_T(t) \) by multiplying it with a comb function \( \text{comb}(t) \):

\[
x(t) \triangleq x_T(t) \text{comb}(t) = x_T(t) \sum_{m=-\infty}^{\infty} \delta(t - mt_0) = \sum_{m=-\infty}^{\infty} x_T(mt_0) \delta(t - mt_0)
\]

As \( x(t) \) is also periodic, its \( n \)th Fourier coefficient is

\[
X[n] = \frac{1}{T} \int_{T} x(t) e^{-j2\pi nf_0 t} dt = \frac{1}{T} \int_{T} \left[ \sum_{m=-\infty}^{\infty} x_T(mt_0) \delta(t - mt_0) \right] e^{-j2\pi nf_0 t} dt
\]

\[\overset{*}{=} \frac{1}{T} \sum_{m=0}^{N-1} x_T(mt_0) \int_{T} \delta(t - mt_0) e^{-j2\pi n f_0 t} dt \]

\[= \frac{1}{T} \sum_{m=0}^{N-1} x_T(mt_0) e^{-j2\pi n f_0 m_0} \]

\[\overset{**}{=} \frac{1}{T} \sum_{m=0}^{N-1} x[m] e^{-j2\pi nm/N} \quad (n = 0, 1, \cdots, N - 1)
\]

Note:

(*) Although there are originally infinite number of terms in the summation, all those corresponding to the impulses outside the integral range \( T \) are zero. Consequently, the summation has only \( N = T/t_0 \) non-zero terms corresponding to those impulses inside the period \( T \).

(**) Here the \( n \)th sample of the signal \( x_T(mt_0) \) is represented by a discrete value \( x[m] \) and \( t_0 f_0 = 1/N \).

To get the inverse transform, we multiply both sides by \( \frac{1}{T} e^{j2\pi mn/N} \), and take summation with respect to \( n \) from 0 to \( N - 1 \) to get

\[
\frac{1}{F} \sum_{n=0}^{N-1} X[n] e^{j2\pi mn/N} = \frac{1}{F} \sum_{n=0}^{N-1} \left[ \frac{1}{T} \sum_{k=0}^{N-1} x[k] e^{-j2\pi nk/N} \right] e^{j2\pi mn/N}
\]

\[= \sum_{k=0}^{N-1} x[k] \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi n[m-k]/N} = \sum_{k=0}^{N-1} x[k] \sum_{l=-\infty}^{\infty} \delta[m - k - lN] \]

\[= \sum_{l=-\infty}^{\infty} x[m - lN] \quad (m = 0, 1, \cdots, N - 1)
\]
Here we have used the fact that
\[
\frac{1}{N} \sum_{n=0}^{N-1} e^{\pm j2\pi nk/N} = \sum_{l=-\infty}^{\infty} \delta[k - lN] = \begin{cases} 
1 & k = lN \\
0 & \text{else}
\end{cases}
\]

which is similar to an equation we obtained before:
\[
\sum_{n=-\infty}^{\infty} e^{j2\pi fn} = \sum_{k=-\infty}^{\infty} \delta(f - k)
\]

**Proof:** If \( k = lN \), then
\[
\frac{1}{N} \sum_{n=0}^{N-1} e^{\pm j2\pi nk/N} = \frac{1}{N} \sum_{n=0}^{N-1} e^{\pm j2\pi nl} = \frac{1}{N} \sum_{n=0}^{N-1} 1 = \frac{N}{N} = 1
\]

If \( k \neq lN \), then
\[
\frac{1}{N} \sum_{n=0}^{N-1} e^{\pm j2\pi nk/N} = \frac{1}{N} \sum_{n=0}^{N-1} (e^{\pm j2\pi k/N})^n = \frac{1 - e^{\pm j2\pi kN/N}}{N(1 - e^{\pm j2\pi k/N})} = 0
\]
We redefine $X[n]$ by dividing its expression above by $F$ and get

\[ X[n] = \frac{1}{N} \sum_{m=0}^{N-1} x[m] e^{-j2\pi mn/N} \quad (n = 0, 1, \cdots, N - 1) \]

and the expression of $x[m]$ becomes

\[ x[m] = \sum_{n=0}^{N-1} X[n] e^{j2\pi mn/N} \quad (m = 0, 1, \cdots, N - 1) \]

**Note 1:** As both $x[m]$ and $X[n]$ are periodic with period $N$, the summation in either the forward or inverse transform does not have to be from 0 to $N - 1$. It can be any $N$ consecutive samples, such as from $-N/2$ to $N/2-1$.

**Note 2:** It is not essential where to put the factor $1/N$. It can also be put in the expression of $x[m]$, or, to make the forward and inverse transforms symmetric, we can put $1/\sqrt{N}$ in both expressions. As both the forward and inverse transforms are discrete and finite, they can be carried out by digital computers with a fast algorithm called Fast Fourier Transform (FFT) to reduce the complexity of the transform from $O(N^2)$ to $O(N \log_2 N)$.

Similar to the spectrum of continuous periodic signals $x_T(t)$:

\[ X(f) = \sum_{k=-\infty}^{\infty} X[k] \delta(f - k f_0) = \sum_{k=-\infty}^{\infty} X[k] \delta(f - \frac{k}{T}) \]

or

\[ X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(e^{j(\omega - k\omega_0)}) = 2\pi \sum_{k=-\infty}^{\infty} X[k] \delta(\omega - k\omega_0) \]

the spectrum $X(f)$ of a discrete periodic signal $x[m]$ $(m = 0, 1, \cdots, N - 1)$ can be written as

\[ X(f) = \sum_{n=-\infty}^{\infty} X[n] \delta(f - \frac{n}{N}) \]

or

\[ X(e^{j\omega}) = 2\pi \sum_{n=-\infty}^{\infty} X[n] \delta(\omega - \frac{2\pi n}{N}) \]

Note that this spectrum is periodic with period $N$, i.e., the $N$ coefficients $X[n]$ $(n = 0, 1, \cdots, N - 1)$ appear repeatedly over the frequency axis, just like the $N$ samples $x[m]$ $(m = 0, 1, \cdots, N - 1)$ of the signal which appear repeatedly over the time axis.
Example 1. Consider a discrete complex exponential with period $N$ ($N$ samples per period)

$$x[m] = e^{jm\omega_0} = e^{j2\pi m f_0} = e^{j2\pi m/N}$$

The Fourier coefficient of this signal is

$$X[n] = \frac{1}{N} \sum_{m=0}^{N-1} x[m] e^{-j2\pi mn/N} = \frac{1}{N} \sum_{m=0}^{N-1} e^{j2\pi m/N} e^{-j2\pi mn/N}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} e^{-j2\pi m(n-1)/N} = \sum_{l=-\infty}^{\infty} \delta[n - 1 - lN]$$

We see that $X[n]$ is 1 when $n = 1, 1 \pm N, 1 \pm 2N, \cdots$.

Example 2. Consider a sinusoidal signal

$$x[m] = \cos(m\frac{2\pi}{5}) = \frac{1}{2} [e^{jm\frac{2\pi}{5}} + e^{-jm\frac{2\pi}{5}}]$$

with $\omega_0 = 2\pi/5$, or period $N = 5$. Its nth Fourier coefficient is:

$$X[n] = \frac{1}{N} \sum_{m=0}^{N-1} x[m] e^{-j2\pi mn/N} = \frac{1}{2} \sum_{m=0}^{4} [e^{-jm2\pi/5} e^{j2\pi mn/5} + e^{-jm2\pi/5} e^{-j2\pi mn/5}]$$

$$= \frac{1}{2} \sum_{l=-\infty}^{\infty} [\delta[n + 1 - lN] + \delta[n - 1 - lN]]$$

Example 3. Consider a symmetric square wave with a period of $N$ samples:

$$x[m] = \begin{cases} 1 & |m| \leq N_1 \\ 0 & N_1 < |m| \leq N/2 \end{cases}$$

For convenience, we choose the limits of the Fourier transform summation from $-N/2$ to $N/2 - 1$, instead of from 0 to $N - 1$ to get

$$X[n] = \frac{1}{N} \sum_{m=-N/2}^{N/2-1} x[m] e^{-j2\pi mn/N} = \frac{1}{N} \sum_{m=-N_1}^{N_1} e^{-j2\pi mn/N}$$

Let $m' = m + N_1$, we have $m = m' - N_1$ and

$$X[n] = \frac{1}{N} \sum_{m'=-0}^{2N_1} e^{-j2\pi m'n/N} e^{j2\pi N_1 n/N}$$

$$= \frac{1}{N} e^{j2\pi N_1 n/N} \frac{1 - e^{-j2\pi(2N_1+1)n/N}}{1 - e^{-j2\pi n/N}}$$

$$= \frac{1}{N} e^{j2\pi N_1 n/N} e^{-j\pi(2N_1+1)n/N} \frac{e^{j\pi(2N_1+1)n/N} + e^{-j\pi(2N_1+1)n/N}}{e^{-j\pi n/N} (e^{j\pi n/N} - e^{-j\pi n/N})}$$

$$= \frac{1}{N} \frac{\sin((2N_1+1)n\pi/N)}{\sin(n\pi/N)}$$

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Properties of Discrete Fourier Transform

Fourier transform of discrete and periodic signals is one of the special cases of general Fourier transform and shares all of its properties discussed earlier. Here we only show some of the properties.

- Convolution theorem

The convolution of two discrete and periodic signal \( x[m] \) and \( y[m] \) \((m = 0, \ldots, N - 1)\) is defined as

\[
x[m] * y[m] \triangleq \sum_{n=0}^{N-1} x[n] y[m - n]
\]

The convolution theorem states:

\[
\mathcal{F}[x[m] * y[m]] = X[n]Y[n] \quad (a)
\]

\[
\mathcal{F}[x[m]y[m]] = X[n] \ast Y[n] \quad (b)
\]

Proof of (a):

\[
\mathcal{F}[x[m] * y[m]] = \frac{1}{N} \sum_{m=0}^{N-1} \left[ \sum_{k=0}^{N-1} x[k] y[m - k] \right] e^{-j2\pi mn/N}
\]

\[
= \frac{1}{N} \sum_{k=0}^{N-1} x[k] \sum_{m=0}^{N-1} y[m - k] e^{-j2\pi (m-k)n/N} e^{-j2\pi kn/N}
\]

\[
= \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-j2\pi kn/N} \sum_{m=0}^{N-1} y[m - k] e^{-j2\pi (m-k)n/N}
\]

\[
= N X[n]Y[n]
\]

Note that due to periodicity of the signal \( y[m - k] \), the second summation is still for the same \( N \) samples over one period, and therefore is the Fourier transform of signal \( y \).
Proof of (b):

\[
\mathcal{F}[x[m]y[m]] = \sum_{m=0}^{N-1} x[m]y[m]e^{-j2\pi mn/N}
\]

\[
= \sum_{m=0}^{N-1} \sum_{k=0}^{N-1} X[k]e^{j2\pi mk/N}y[m]e^{-j2\pi mn/N}
\]

\[
= \sum_{k=0}^{N-1} X[k] \sum_{m=0}^{N-1} y[m]e^{-j2\pi m(n-k)/N}
\]

\[
= \sum_{k=0}^{N-1} X[k]Y[n-k] = X[n] * Y[n]
\]

• Parseval’s relation

\[
\frac{1}{N} \sum_{m=0}^{N-1} |x[m]|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |X[n]|^2
\]
Four different forms of Fourier transform

To summarize the previously discussed Fourier series and transform of periodic and discrete signals, we see that they can be considered as different forms or special cases of the same Fourier transform, and therefore be unified as shown below.

- **I. Non-periodic continuous signal, continuous, non-periodic spectrum**
  This is the most general form of continuous time Fourier transform. The signal is continuous and non-periodic, and so is its spectrum.

\[
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt
\]

\[
x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df
\]

The first one is the forward transform, and the second one is the inverse transform. The Laplace transform can be considered as a modified version of this most general form of Fourier transform.

Alternative expression:

\[
X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt
\]

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j\omega t} \, d\omega
\]

- **II. Non-periodic discrete signal, continuous periodic spectrum**
  The discrete time Fourier transform. The signal can be considered as a sequence of samples of a continuous time function. The time interval between two consecutive samples \(x[m]\) and \(x[m+1]\) is \(t_0 = 1/F\), where \(F\) is the sampling rate, which is also the period of the spectrum in the frequency domain. The discrete time function can be written as

\[
x(t) = \sum_{m=-\infty}^{\infty} x[m] \delta(t - mt_0)
\]

and its transform is:

\[
X_F(f) = \sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi f m t_0}
\]

and the inverse transform is:

\[
x[m] = \frac{1}{F} \int_{F} X_F(f) e^{j2\pi f m t_0} \, df \quad (m = 0, \pm 1, \pm 2, \cdots)
\]

We can verify that the spectrum is indeed periodic:

\[
X_F(f + kF) = \sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi (f + k/t_0) m t_0} = X_F(f) \quad (k = \pm 1, \pm 2, \cdots)
\]

as \(e^{j2\pi m k} = 1\).

Alternative expression:

\[
X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m}
\]

\[
x[m] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega m} \, d\omega
\]

This can be considered as the Z-transform evaluated at \(z = e^{j\omega}\):

\[
X(e^{j\omega}) = X(z) |_{z = e^{j\omega}}
\]
III. Periodic continuous signal, discrete non-periodic spectrum

This is the Fourier series expansion of periodic signals. The time period is $T$, and the interval between two consecutive frequency components in the Fourier domain is $f_0 = 1/T$, and its transform is:

$$X[n] = \frac{1}{T} \int_{T} x_T(t) e^{-j2\pi fn_0 t} dt \quad (n = 0, \pm 1, \pm 2, \cdots)$$

and the inverse transform is:

$$x_T(t) = \sum_{n=-\infty}^{\infty} X[n] e^{j2\pi fn_0 t}$$

The discrete spectrum can also be represented as:

$$X(f) = \sum_{n=-\infty}^{\infty} X[n] \delta(f - nf_0)$$

We can verify that the time function is indeed periodic:

$$x_T(t + kT) = \sum_{n=-\infty}^{\infty} X[n] e^{-j2\pi fn_0 (t+k)/f_0} = x_T(t) \quad (k = \pm 1, \pm 2, \cdots)$$

Alternative expression:

$$a_k = X[k] = \frac{1}{T} \int_{T} x(t) e^{-j\omega_0 t} dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t}$$

IV. Periodic discrete signal, discrete periodic spectrum

This is the discrete Fourier transform (DFT). As both the signal and its spectrum are discrete and periodic (and therefore finite as only one period is needed in the computation), this is the only form of Fourier transform that can be actually carried out by a digital computer. Moreover, due to the Fast Fourier Transform (FFT) algorithm, the computational complexity of the DFT can be reduced from $O(N^2)$ to $O(N\log_2 N)$ ($N = T/t_0 = F/f_0$ being the number of samples in both temporal and frequency period).

$$X[n] = \frac{1}{T} \sum_{m=0}^{N-1} x[m] e^{-j2\pi mn_0 f_0} \quad (n = 0, 1, \cdots, N - 1)$$

$$x[m] = \frac{1}{F} \sum_{n=0}^{N-1} X[n] e^{j2\pi mn_0 f_0} \quad (m = 0, 1, \cdots, N - 1)$$

where $N$ is the number of samples in the period $T$, which is also the number of frequency components in the spectrum:

$$N = \frac{T}{t_0} = \frac{1/f_0}{1/F} = \frac{F}{f_0}$$

We therefore also have $TF = N$ and $t_0 f_0 = 1/N$. The DFT can be redefined as

$$X[n] = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} x[m] e^{-j2\pi mn/N} = \sum_{m=0}^{N-1} w_{-}^{mn} x[m] \quad (n = 0, 1, \cdots, N - 1)$$

$$x[m] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X[n] e^{j2\pi mn/N} = \sum_{n=0}^{N-1} w_{-}^{-mn} X[n] \quad (m = 0, 1, \cdots, N - 1)$$

where $w_{-} \triangleq e^{-j2\pi/N}/\sqrt{N}$. We can easily verify that the time function and its spectrum are indeed periodic: $x[m + kN] = x[m]$ and $X[n + kN] = X[n]$. 
The four forms of Fourier transform can be summarized as below:

<table>
<thead>
<tr>
<th></th>
<th>The signal $x(t)$</th>
<th>The spectrum $X(f)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Continuous, Non-periodic $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$</td>
<td>Non-periodic, Continuous $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$</td>
</tr>
<tr>
<td>II</td>
<td>Discrete $(t_0)$, Non-periodic $x(t) = \sum_{m=-\infty}^{\infty} x[m] \delta(t - m t_0)$ $x[m] = \sum_{F} X_F(f) e^{j2\pi f m t_0} df/F$</td>
<td>Periodic $(F = 1/t_0)$, Continuous $X_F(f) = \sum_{m=-\infty}^{\infty} x[m] e^{-j2\pi f m t_0}$</td>
</tr>
<tr>
<td>III</td>
<td>Continuous, Periodic $(T)$ $x_T(t) = \sum_{n=-\infty}^{\infty} X[n] e^{j2\pi n f t}$</td>
<td>Non-periodic, Discrete $(f_0 = 1/T)$ $X[n] = \int_{T} x_T(t) e^{-j2\pi n f t} dt/T$ $X(f) = \sum_{n=-\infty}^{\infty} X[n] \delta(f - n f_0)$ $F/f_0 = T/t_0 = N$</td>
</tr>
<tr>
<td>IV</td>
<td>Discrete $(t_0)$, Periodic $(T)$ $x_T[m] = \sum_{m=0}^{N-1} X[n] e^{j2\pi m n / N}$ $x_T(t) = \sum_{m=0}^{N-1} x[m] \delta(t - m t_0)$ $X_T[n] = \sum_{m=0}^{N-1} x[m] e^{-j2\pi m n / N}$ $X_T[f] = \sum_{n=0}^{N-1} X[n] \delta(f - n f_0)$ $F/f_0 = T/t_0 = N$</td>
<td>Periodic $(F = 1/t_0)$, Discrete $(f_0 = 1/T)$ $X_T[n] = \sum_{m=0}^{N-1} x[m] e^{-j2\pi m n / N}$ $X_T[f] = \sum_{n=0}^{N-1} X[n] \delta(f - n f_0)$ $F/f_0 = T/t_0 = N$</td>
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</table>

All four forms of the Fourier transform share the same properties discussed above mostly for the continuous and non-periodic case, although these properties may take different forms for each of the four cases. Here only the convolution theorem and Parseval’s formula are summarized.

The Convolution theorem

<table>
<thead>
<tr>
<th></th>
<th>Time domain</th>
<th>Frequency domain</th>
</tr>
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<tbody>
<tr>
<td>I</td>
<td>$x(t)y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$ $x(t) \cdot y(t)$</td>
<td>$X(f) \cdot Y(f)$ $X(f) * Y(f) = \int_{-\infty}^{\infty} X(\phi)y(f - \phi)d\phi$</td>
</tr>
<tr>
<td>II</td>
<td>$x_T(t) \cdot y_T(\tau) = \frac{1}{T} \int_{T} x_T(\tau)y_T(t-\tau)d\tau$ $x_T(t) \cdot y_T(t)$</td>
<td>$X[n] \cdot Y[n]$ $X[n] * Y[n] = \sum_{k=-\infty}^{\infty} X[k]Y[n-k]$</td>
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<tr>
<td>III</td>
<td>$x[m] \cdot y[m] = \sum_{k=-\infty}^{\infty} x[k]y[m-k]$ $x[m] \cdot y[m]$</td>
<td>$X_F(f) \cdot Y_F(f)$ $X_F(f) * Y_F(f) = \frac{1}{T} \int_{T} X_F(\phi)Y_F(f - \phi)d\phi$</td>
</tr>
<tr>
<td>IV</td>
<td>$x_T[m] \cdot y_T[m] = \sum_{k=0}^{N-1} x[k]y[m-k]$ $x_T[m] \cdot y_T[m]$</td>
<td>$X_F[n] \cdot Y_F[n]$ $X_F[n] * Y_F[n] = \sum_{k=0}^{N-1} X_F[k]Y_F[n-k]$</td>
</tr>
</tbody>
</table>
The Parseval’s formula

\begin{itemize}
  \item I \hspace{2cm} \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df
  \item II \hspace{2cm} \frac{1}{T} \int_{T} |x_T(t)|^2 dt = \sum_{n=-\infty}^{\infty} |X[n]|^2
  \item III \hspace{2cm} \sum_{m=-\infty}^{\infty} |x[m]|^2 = \frac{1}{F} \int_{F} |X_F(f)|^2 df
  \item IV \hspace{2cm} \frac{1}{T} \sum_{m=0}^{N-1} |x_T[m]|^2 = \frac{1}{F} \sum_{n=0}^{N-1} |X_F[n]|^2
\end{itemize}