Digital Convolution — E186 Handout

The convolution of two continuous signals is defined as

\[ y(t) = h(t) \ast x(t) \triangleq \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) d\tau \]

i.e., convolution operation is commutative. Also it is associative:

\[ h \ast (g \ast x) = (h \ast g) \ast x \]

As a typical example, \( y(t) \) is the output of a system characterized by its impulse response function \( h(t) \) with input \( x(t) \).

Convolution in discrete form is

\[ y(n) = \sum_{m=-\infty}^{\infty} x(n - m) h(m) = \sum_{m=-\infty}^{\infty} h(n - m) x(m) \]

If \( h(m) \) is finite, i.e.,

\[ h(m) = \begin{cases} h(m) & |m| \leq k \\ 0 & |m| > k \end{cases} \]

the convolution becomes

\[ y(n) = \sum_{m=-k}^{k} x(n - m) h(m) \]

In time domain, all realistic systems are causal

\[ y(n) = 0 \quad \text{if } n < 0 \]

However, in image processing, we often consider convolution in spatial domain where causality does not apply.
If \( h(m) \) is symmetric (almost always true in image processing), i.e.,
\[
h(-m) = h(m)
\]
the convolution becomes
\[
y(n) = \sum_{m=-k}^{k} x(n+m) h(m)
\]
If \( x(m) \) is also finite (always true in reality), i.e.,
\[
x(m) = \begin{cases} 
  x(m) & 0 \leq m < N \\
  0 & \text{otherwise}
\end{cases}
\]
for \( x(n+m) \) to be in the valid non-zero range, its index \((n+m)\) has to satisfy:
\[
0 \leq (n+m) \leq N - 1
\]
correspondingly for \( y(n) \) to be non-zero, its index \( n \) has to satisfy:
\[
-k \leq n \leq N - m - 1
\]
When \( m = k \), the lower bound \( n = -k \) is reached, and when \( m = -k \), the upper bound \( n = N + k - 1 \) is reached. In other words, there are \( N + 2k \) valid elements in the output:
\[
y(n), \quad (-k \leq n \leq N + k - 1)
\]
This convolution can be best understood graphically (where the index of \( y(n) \) is rearranged).

In image processing, all the discussions above for one-dimensional convolution are generalized into two dimensions, and \( h \) is called a convolution kernel.