Fourier Transform — E180 Handout

Four different forms of the Fourier transform

- **Non-periodic, continuous time function** $x(t)$, continuous, non-periodic spectrum $X(f)$
  
  This is the most general form of Fourier transform.

  $$
  X(f) = \int_{-\infty}^{+\infty} x(t)e^{-j2\pi ft}\,dt
  $$

  $$
  x(t) = \int_{-\infty}^{+\infty} X(f)e^{j2\pi ft}\,df
  $$

  The first one is the forward transform, and the second one is the inverse transform.

- **Non-periodic, discrete time function** $x(n)$, continuous, periodic spectrum $X_F(f)$

  The discrete time function can be considered as a sequence of samples of continuous time function. The time interval between two consecutive samples $x(m)$ and $x(m+1)$ is $t_0 = 1/F$, where $F$ is the sampling rate, which is also the period of the spectrum in the frequency domain.

  The discrete time function can be written as

  $$
  x(t) = \sum_{m=-\infty}^{+\infty} x(m)\delta(t - mt_0)
  $$

  and its transform is:

  $$
  X_F(f) = \sum_{m=-\infty}^{+\infty} x(m)e^{-j2\pi f m t_0}
  $$

  $$
  x(m) = \frac{1}{F} \int_{-F/2}^{+F/2} X_F(f)e^{j2\pi f m t_0}\,df
  $$

  $(m = 0, \pm 1, \pm 2, \cdots)$
We can verify that the spectrum is indeed periodic:

\[ X_F(f + kF) = X_F(f + k/f_0) = \sum_{m=-\infty}^{+\infty} x(m)e^{-j2\pi(f+k/f_0)mt_0} = X_F(f) \]

(for \( k = \pm 1, \pm 2, \cdots \)) because \( e^{\pm j2\pi mk} = 1 \).

**Periodic, continuous time function** \( x_T(t) \), **discrete, non-periodic spectrum** \( X(n) \)

This is the Fourier series expansion of periodic functions. The time period is \( T \), and the interval between two consecutive frequency components is \( f_0 = 1/T \), and its transform is:

\[
X(n) = \frac{1}{T} \int_{-T/2}^{+T/2} x_T(t)e^{-j2\pi fn_0t} dt
\]

\[
x_T(t) = \sum_{n=-\infty}^{+\infty} X(n)e^{j2\pi fn_0t}
\]

\[ n = 0, \pm 1, \pm 2, \cdots \]

The discrete spectrum can also be represented as:

\[
X(f) = \sum_{n=-\infty}^{+\infty} X(n)\delta(f - nf_0)
\]

We can verify that the time function is indeed periodic:

\[
x_T(t + kT) = x_T(t + k/f_0) = \sum_{n=-\infty}^{+\infty} X(n)e^{-j2\pi fn_0(t+k/f_0)} = x_T(t)
\]

(for \( k = \pm 1, \pm 2, \cdots \))
• **Periodic, discrete time function** $x(m)$, **discrete, periodic spectrum** $X(n)$

This is the discrete Fourier transform (DFT).

$$X(n) = \frac{1}{T} \sum_{m=0}^{M-1} x(m)e^{-j2\pi nm/T_0}$$

$$n = 0, 1, \ldots, M - 1$$

$$x(m) = \frac{1}{F} \sum_{n=0}^{M-1} X(n)e^{j2\pi nm/F_0}$$

$$m = 0, 1, \ldots, M - 1$$

where $M$ is the number of samples in the period $T$, which is also the number of frequency components in the spectrum:

$$M = \frac{T}{t_0} = \frac{1}{f_0} = \frac{F}{f_0}$$

We therefore also have $TF = M$ and $t_0f_0 = 1/M$.

The DFT can be redefined as

$$X(n) = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} x(m)e^{-j2\pi mn/M} = \sum_{m=0}^{M-1} w_M^{mn} x(m)$$

$$n = 0, 1, \ldots, M - 1$$

$$x(m) = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} X(n)e^{j2\pi mn/M} = \sum_{n=0}^{M-1} w_M^{-mn} X(n)$$

$$m = 0, 1, \ldots, M - 1$$

where $w_M \triangleq e^{-j2\pi/M}/\sqrt{M}$.

We can easily verify that the time function and its spectrum are indeed periodic: $x(m + kM) = x(m)$ and $X(n + kM) = X(n)$.
The $\delta$ function

The discrete and periodic time function and spectrum can be written as, respectively

$$x_T(t) = \sum_{m=-\infty}^{+\infty} x(m)\delta(t - mt_0)$$

$$X_F(f) = \sum_{n=-\infty}^{+\infty} X(n)\delta(f - nf_0)$$

The $\delta$ function used above satisfies the following:

1. $$\delta(t - \tau) = \begin{cases} 0 & t \neq \tau \\ \infty & t = \tau \end{cases}$$

2. $$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$

3. $$\int_{-\infty}^{+\infty} x(t)\delta(t - \tau)\,dt = x(\tau)$$
Vector form of 1D-DFT

The above summation expression for DFT can also be written in more convenient form of matrix-vector multiplication:

$$\begin{bmatrix}
X(0) \\
\vdots \\
X(M-1)
\end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix}
e^{-j2\pi Mn} \\
\vdots \\
e^{-j2\pi Mn}
\end{bmatrix} \begin{bmatrix}
x(0) \\
\vdots \\
x(M-1)
\end{bmatrix}$$

and

$$\begin{bmatrix}
x(0) \\
\vdots \\
x(M-1)
\end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix}
e^{-j2\pi Mn} \\
\vdots \\
e^{-j2\pi Mn}
\end{bmatrix} \begin{bmatrix}
X(0) \\
\vdots \\
X(M-1)
\end{bmatrix}$$

It is obvious that the complexity of 1D DFT takes is $O(N^2)$, which, as we will see later, can be reduced to $O(N\log_2 N)$ by Fast Fourier Transform (FFT) algorithms.

These matrix-vector multiplications can be represented more concisely as:

$$\mathbf{X} = \mathbf{W}^{-1}\mathbf{\bar{x}}$$

and

$$\mathbf{\bar{x}} = \mathbf{W}\mathbf{X}$$

where both $\mathbf{X}$ and $\mathbf{\bar{x}}$ are $M \times 1$ column (vertical) vectors:

$$\mathbf{X} \triangleq \begin{bmatrix}
X(0) \\
\vdots \\
X(M-1)
\end{bmatrix}_{M \times 1}$$

$$\mathbf{\bar{x}} \triangleq \begin{bmatrix}
x(0) \\
\vdots \\
x(M-1)
\end{bmatrix}_{M \times 1}$$

and $\mathbf{W}$ is an $M \times M$ matrix:

$$\mathbf{W} = \begin{bmatrix}
\vdots & \vdots & \vdots \\
\cdot & w_{mn} & \cdot \\
\cdot & \cdot & \cdot \\
\end{bmatrix}_{M \times M}$$
where \( w_{mn} \) is an element in the \( m \)th row and \( n \)th column of matrix \( W \) defined as
\[
  w_{mn} \triangleq \frac{1}{\sqrt{M}} (e^{j2\pi/M})^{mn}
\]
whose complex conjugate is
\[
  w_{mn}^* = \frac{1}{\sqrt{M}} (e^{-j2\pi/M})^{mn}
\]

Obviously \( W \) is symmetric (\( w_{mn} = w_{nm} \))
\[
  W^T = W
\]
but \( W \) is not Hermitian:
\[
  W^{*T} = W^* \neq W
\]
\( W \) is a unitary matrix,
\[
  W^{*T} = W^* = W^{-1}
\]
because its rows (or columns) are orthogonal:
\[
(W_m, W_{m'}) = \sum_{k=1}^{M} w_{mk}^* w_{m'k} = \frac{1}{M} \sum_{k=1}^{M} (e^{-j2\pi/M})^{mk} (e^{j2\pi/M})^{m'k} = \frac{1}{M} \sum_{k=1}^{M} (e^{j2\pi/M})^{(m'-m)k} = \delta_{m'm}
\]
(* Why?)
The DFT pair can be rewritten as:
\[
\mathbf{X} = W^* \mathbf{x}
\]
\[
\mathbf{x} = W \mathbf{X}
\]
Fast Fourier Transform (FFT) Algorithm

The M-point DFT of time samples \(x(0), x(1), \ldots, x(M - 1)\) is defined as (ignoring the coefficient \(1/\sqrt{M}\) for now):

\[
X(n) = \sum_{m=0}^{M-1} x(m)e^{-j2\pi mn/M} = \sum_{m=0}^{M-1} x(m)w_{M}^{mn}
\]

for

\(n = 0, 1, \ldots, M - 1\)

\(w_{M}\) is defined as \(w_{M} \triangleq e^{-j2\pi/M}\) and it is easy to show that \(w_{M}\) has the following properties:

1. \(w_{M}^{kM} \equiv 1\)
2. \(w_{2M}^{2k} \equiv w_{M}^{k}\)
3. \(w_{2M}^{M} \equiv -1\)

Let \(M = 2N\), the above DFT can be written as

\[
X(n) = \sum_{m=0}^{N-1} x(2m)w_{2N}^{mn} + \sum_{m=0}^{N-1} x(2m + 1)w_{2N}^{(2m+1)n}
\]

The first summation has all the even terms and the second all the odd ones. Due to the 2nd property of \(w_{M}\), the above can be rewritten as

\[
X(n) = \sum_{m=0}^{N-1} x(2m)w_{N}^{mn} + \sum_{m=0}^{N-1} x(2m + 1)w_{N}^{mn}w_{2N}^{n}
\]

We define

\[
X_{even}(n) \triangleq \sum_{m=0}^{N-1} x(2m)w_{N}^{mn}
\]

and

\[
X_{odd}(n) \triangleq \sum_{m=0}^{N-1} x(2m + 1)w_{N}^{mn}
\]

They are \(M/2\)-point DFTs. The original M-point DFT becomes

\[
X(n) = X_{even}(n) + X_{odd}(n)w_{2N}^{n}\tag{1}
\]
Here we let the index $n$ cover only the first half of the original range of the DFT, $n = 0, 1, \ldots, M/2 - 1 = N - 1$. The second half can be obtained by replacing $n$ in Eq. (1) by $n + N$:

$$X(n + N) = X_{\text{even}}(n + N) + X_{\text{odd}}(n + N)w_{2N}^{n+N}$$

Due to the first property of $w_M$, we have

$$X_{\text{even}}(n + N) = \sum_{m=0}^{N-1} x(2m)w_N^{m(n+N)} = \sum_{m=0}^{N-1} x(2m)w_N^{mn} = X_{\text{even}}(n)$$

and similarly

$$X_{\text{odd}}(n + N) = X_{\text{odd}}(n)$$

Also, due to the 3rd property of $w_M$, we have

$$w_{2N}^{n+N} = w_{2N}^n w_{2N}^N = -w_{2N}^n$$

Now the second half of the DFT becomes

$$X(n + N) = X_{\text{even}}(n) - X_{\text{odd}}(n)w_{2N}^n$$

(2)

The M-point DFT can now be obtained from Eqs. (1), (2), once $X_{\text{even}}(n)$ and $X_{\text{odd}}(n)$ are available. However, since $X_{\text{even}}(n)$ and $X_{\text{odd}}(n)$ are $M/2$-point DFTs, they can be obtained the same way. This process goes on recursively until finally only 1-point DFTs are needed, which are just the time samples themselves. Therefore, the operations of an M-point DFT can be symbolically represented by the following diagram. The complexity is therefore reduced from $O(M^2)$ to $O(M \log_2 M)$. 
Fourier Transform 2 Real Functions with 1 DFT

First we recall the symmetry properties of the DFT. The DFT of \( x(m) = x_r(m) + jx_i(m) \) is defined as

\[
X(n) = \sum_{m=0}^{M-1} x(m) e^{-j2\pi mn/M} \\
= \sum_{m=0}^{M-1} x_r(m) e^{-j2\pi mn/M} + j \sum_{m=0}^{M-1} x_i(m) e^{-j2\pi mn/M} \\
= \sum_{m=0}^{M-1} x_r(m) \cos(2\pi mn/M) - j \sum_{m=0}^{M-1} x_r(m) \sin(2\pi mn/M) \\
+ j[\sum_{m=0}^{M-1} x_i(m) \cos(2\pi mn/M) - j \sum_{m=0}^{M-1} x_i(m) \sin(2\pi mn/M)] \\
= \sum_{m=0}^{M-1} [x_r(m) \cos(2\pi mn/M) + x_i(m) \sin(2\pi mn/M)] \\
+ j \sum_{m=0}^{M-1} [x_i(m) \cos(2\pi mn/M) - x_r(m) \sin(2\pi mn/M)] \\
= X_r(n) + jX_i(n)
\]

where \( X_r(n) \) and \( X_i(n) \) are the real and imaginary part of the spectrum respectively. If \( x(m) \) is real, i.e., \( x_i(m) \equiv 0 \), then we have

\[
\begin{aligned}
X_r(-n) &= X_r(n) \\
X_i(-n) &= -X_i(n)
\end{aligned}
\]

or

\[
X(-n) = X_r(-n) + jX_i(-n) = X_r(n) - jX_i(n) = X^*(n)
\]

If \( x(m) \) is imaginary, i.e., \( x_r(m) \equiv 0 \), then we have

\[
\begin{aligned}
X_r(-n) &= -X_r(n) \\
X_i(-n) &= X_i(n)
\end{aligned}
\]

or

\[
X(-n) = X_r(-n) + jX_i(-n) = -X_r(n) + jX_i(n) = -X^*(n)
\]
Next we show how an arbitrary function \( f(x) \) can be decomposed into the even and odd components \( f_e(x) \) and \( f_o(x) \):
\[
\begin{align*}
  f_e(x) &= (f(x) + f(-x))/2 \\
  f_o(x) &= (f(x) - f(-x))/2
\end{align*}
\]
and
\[
f_e(x) + f_o(x) = f(x)
\]

Now we are ready to show how to Fourier transform two real functions \( x_1(m) \) and \( x_2(m) \) to get their spectra \( X_1(n) \) and \( X_2(n) \) by one DFT.

1. Define a complex function \( x(m) \) by the two real functions:
\[
x(m) \equiv x_1(m) + jx_2(m)
\]
Notice here that we impose \( j \) on \( x_2(m) \) to make it imaginary.

2. Find the DFT of \( x(m) \)
\[
DFT[x(m)] = X(n) = X_r(n) + jX_i(n)
\]

3. Separate \( X(n) \) into \( X_1(n) \) and \( X_2(n) \), the spectra of \( x_1(m) \) and \( x_2(m) \), using the symmetry properties discussed previously.

- Since \( x_1(m) \) is real, the real part of its spectrum \( X_1(n) \) is the even component of \( X_r(n) \) and the imaginary part of \( X_1(n) \) is the odd component of \( X_i(n) \), i.e.,
\[
X_1(n) = X_{1r}(n) + jX_{1i}(n) = \frac{X_r(n) + X_r(-n)}{2} + j\frac{X_i(n) - X_i(-n)}{2}
\]

- Since \( jx_2(m) \) is imaginary, the real part of its spectrum \( jX_2(n) \) is the odd component of \( X_r(n) \) and the imaginary part of \( jX_2(n) \) is the even component of \( X_i(n) \), i.e.,
\[
jX_2(n) = j[X_{2r}(n) + jX_{2i}(n)] = \frac{X_r(n) - X_r(-n)}{2} + j\frac{X_i(n) + X_i(-n)}{2}
\]
Dividing both sides by \( j \), we get
\[
X_2(n) = X_{2r}(n) + jX_{2i}(n) = \frac{X_i(n) + X_i(-n)}{2} - j\frac{X_r(n) - X_r(-n)}{2}
\]
Note that \( X(-n) = X(N - n) \) because \( X(n) \) is a periodic function.
Two-Dimensional Fourier Transform (2D-FT)

Similar to 1D-FT, 2D-FT can also have four different forms depending on whether the 2D signal (usually spatial signal) $f(x, y)$ is periodic and whether it is discrete. Here we consider only two cases:

- **2D Fourier transform pair of a Non-periodic, continuous signal** $f(x, y)$ is

  $$ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} dx \, dy $$

  $$ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} du \, dv $$

  where $u$ and $v$ are spatial frequencies in $x$ and $y$ directions, respectively, and $F(u, v)$ is the 2D spectrum of $f(x, y)$.

- **2D discrete Fourier transform pair of a finite (periodic) and discrete signal** $x(m, n)$, $(0 \leq m \leq M - 1, 0 \leq n \leq N - 1)$ is

  $$ X(k, l) = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} x(m, n) e^{-j2\pi\left(\frac{mk}{M} + \frac{nl}{N}\right)} $$

  $$ x(m, n) = \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} X(k, l) e^{j2\pi\left(\frac{mk}{M} + \frac{nl}{N}\right)} $$

  $(0 \leq m, k \leq M - 1, 0 \leq n, l \leq N - 1)$

  where $M$ and $N$ are the numbers of samples in $x$ and $y$ directions, respectively, and $X(k, l)$ is the 2D discrete spectrum of $x(m, n)$. Both $X(k, l)$ and $x(m, n)$ can be considered as elements in two $M$ by $N$ matrices $[x]$ and $[X]$, respectively.

**Example 1**

$$ f(x, y) = \begin{cases} 
1 & \text{if } \left(-\frac{a}{2} < x < \frac{a}{2}, -\frac{b}{2} < y < \frac{b}{2}\right) \\
0 & \text{else} 
\end{cases} $$
\[ F(u,v) = \int \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} \, dx \, dy \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j2\pi ux/2} \, dx \int_{-\infty}^{\infty} e^{-j2\pi vy/2} \, dy \]
\[ = \frac{\sin(\pi u)}{\pi u} \frac{\sin(\pi v)}{\pi v} \]

See Fig. 3.2 on page 85 of the textbook.

**Example 2**

\[ f(x,y) = \begin{cases} 
1 & x^2 + y^2 < R^2 \\
0 & \text{else} 
\end{cases} \]

It is more convenient to use polar coordinate systems in both spatial and frequency domains. Let

\[ \begin{align*} 
x &= r \cos \theta, & y &= r \sin \theta \\
\rho &= \sqrt{x^2 + y^2}, & \phi &= \tan^{-1}(y/x) \\
\end{align*} \]

\[ dx \, dy = r \, dr \, d\theta \]

and

\[ \begin{align*} 
u &= \rho \cos \phi, & v &= \rho \sin \phi \\
\rho &= \sqrt{u^2 + v^2}, & \phi &= \tan^{-1}(v/u) \\
\end{align*} \]

\[ du \, dv = \rho \, d\rho \, d\phi \]

we have:

\[ F(u, v) = \int \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} \, dx \, dy \]
\[ = \int_{0}^{R} \left[ \int_{0}^{2\pi} e^{-j2\pi \rho (\cos \phi \cos \theta + \sin \phi \sin \theta)} \, d\theta \right] r \, dr \]
\[ = \int_{0}^{R} \left[ \int_{0}^{2\pi} e^{-j2\pi \rho \cos(\theta-\phi)} \, d\theta \right] r \, dr \]
\[ = \int_{0}^{R} \left[ \int_{0}^{2\pi} e^{-j2\pi \rho \cos \theta} \, d\theta \right] r \, dr \]
To continue, we need to use 0th order Bessel function \( J_0(x) \) defined as

\[
J_0(x) \triangleq \frac{1}{2\pi} \int_0^{2\pi} e^{-jx\cos\theta} d\theta
\]

which is related to the 1st order Bessel function \( J_1(x) \) by

\[
\frac{d}{dx}(x J_1(x)) = x J_0(x)
\]

i.e.,

\[
\int_0^x x J_0(x) dx = x J_1(x)
\]

Substituting \( 2\pi r \rho \) for \( x \), we have

\[
F(u, v) = F(\rho, \phi) = \int_0^R 2\pi r J_0(2\pi r \rho) dr
\]

\[
= \frac{1}{\rho} R J_1(2\pi \rho R)
\]

We see that the spectrum \( F(u, v) = F(\rho, \phi) \) is independent of angle \( \phi \) and therefore is central symmetric. See the top example in Fig. 3.3 on page 86 of the textbook.
Matrix Form of 2D DFT

Reconsider the 2D DFT:

\[
X(k, l) = \frac{1}{\sqrt{MN}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} x(m, n) e^{-j2\pi \frac{mk}{M}} e^{-j2\pi \frac{nl}{N}}
\]

\[
= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X'(k, n) e^{-j2\pi \frac{nk}{N}}
\]

\[
(0 \leq m, k \leq M - 1, \ 0 \leq n, l \leq N - 1)
\]

where

\[
X'(k, n) \triangleq \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} x(m, n) e^{-j2\pi \frac{mk}{M}}
\]

As the summation is with respect to the row index \(m\) and the column index \(n\) can be treated as a fixed parameter, this expression can be considered as the Fourier transform of the \(n\)th column of \([x]\), which can be written in column vector (vertical) form as:

\[
\mathbf{X}'_n = W^* \mathbf{x}_n
\]

for all columns \(n = 0, \cdots, N - 1\).

Putting all these \(N\) columns together, we can write

\[
[\mathbf{X}'_0, \cdots, \mathbf{X}'_{N-1}] = W^* [\mathbf{x}_0, \cdots, \mathbf{x}_{N-1}]
\]

or more concisely

\[
[X'] = W^* [x]
\]

where \(W^*\) is a \(N\) by \(N\) Fourier transform matrix.

We then notice that the summation expression for \(X(k, l)\) is with respect to the column index \(n\) and the row index number \(k\) can be treated as a fixed parameter, the expression is the Fourier transform of the \(k\)th row, which can be written in row vector (horizontal) form as

\[
\mathbf{X}'_k = (W^* \mathbf{X}'_k)^T = \mathbf{X}'_k^T W^* T, \quad (k = 0, \cdots, M - 1)
\]
Putting all these $M$ rows together, we can write

$$
\begin{bmatrix}
\overline{X}_0^T \\
\vdots \\
\overline{X}_{M-1}^T
\end{bmatrix}
= 
\begin{bmatrix}
\overline{X}_0^T \\
\vdots \\
\overline{X}_{M-1}^T
\end{bmatrix} W^*
$$

($W$ is symmetric: $W^{*T} = W^*$), or more concisely

$$
[X] = [X'] W^*
$$

Substituting $[X']$ by $W^* [x]$, we have

$$
[X] = W^* [x] W^*
$$

This transform expression indicates that 2D DFT can be implemented by transforming all the rows of $[x]$ and then transforming all the columns of the resulting matrix. The order of the row and column transforms is not important.

Similarly, the inverse 2D DFT can be written as

$$
[x] = W [X] W
$$

Again note that $W$ is a symmetric Unitary matrix:

$$
W^{-1} = W^{*T} = W^*
$$

It is obvious that the complexity of 2D DFT is $O(M^3)$ (assuming $M = N$), which can be reduced to $O(M^2 \log_2 M)$ if FFT is used.